or

$$a^2 + c^2 = b^2 \text{ [a perfect square } - 2] = b^2 k,$$

where k must have one of the values 2 (*i. e.*, $2^2 - 2$), 7, 14, 23, 34, 47, 62, 79, 98, $\cdots$.

If k has one of the values 2, 34, 98, this equation possesses a solution, but not for any of the other values given. To show this and the method of obtaining the solution, it will be sufficient to treat the values $k = 2$, and $k = 7$. If $k = 7$, a and c must be prime to each other, and also to 7, as otherwise they would not be prime to b. Consequently each must be of the form $7\lambda \pm 1$, 2, or 3. But the sum of their squares cannot be divisible by 7 in that case, and the equation is impossible. If $k = 2$, let $x = a/b$ and $y = c/b$, so that the equation takes the form $x^2 + y^2 = 2$. An obvious solution is $x = 1$, $y = 1$. Any line through the point (1, 1) with rational slope will cut the circle $x^2 + y^2 = 2$ in a second rational point, and, conversely, taking the slope to be α/β, where α and β are integers, and solving for x and y, we get the general solution of our equation to be

$$x = \frac{a}{b} = \frac{\alpha^2 - 2\alpha\beta - \beta^2}{\alpha^2 + \beta^2}, \qquad y = \frac{c}{b} = \frac{\beta^2 - 2\alpha\beta - \alpha^2}{\alpha^2 + \beta^2},$$

where α and β are any integers. m and n are the reciprocals of x and y respectively.

A similar solution can readily be found in the other possible cases.

QUESTIONS AND DISCUSSIONS.

[Send all Communications to U. G. MITCHELL, University of Kansas, Lawrence, Kans.]

DISCUSSIONS.

I. Relating to Some Determinants Connected with the Bernoulli Numbers.

By K. P. Williams, Indiana University.

It is well known that the Bernoulli numbers B_1, B_2, B_3, $\cdots$ can be obtained from either of the following recurrence relations,

$${}_{2n+1}C_{2n}B_n - {}_{2n+1}C_{2n-2}B_{n-1} + \cdots (-1)^{n-1}_{2n+1}C_2B_1 = (-1)^{n-1}(n - \tfrac{1}{2}),$$

or

$${}_{2n+2}C_{2n}B_n - {}_{2n+2}C_{2n-2}B_{n-1} + \cdots (-1)^{n-1}_{2n+2}C_2B_1 = (-1)^{n-1}n,$$

where ${}_nC_r$ denotes the combinations of n things r at a time.[1] If we write out several of the equations we obtain the two series of relations

$$\begin{array}{ll}
{}_3C_2B_1 = \tfrac{1}{2}, & {}_4C_2B_1 = 1, \\
{}_5C_2B_1 - {}_5C_4B_2 = \tfrac{3}{2}, & {}_6C_2B_1 - {}_6C_4B_2 = 2, \\
{}_7C_2B_1 - {}_7C_4B_2 + {}_7C_6B_3 = \tfrac{5}{2}, & {}_8C_2B_1 - {}_8C_4B_2 + {}_8C_6B_3 = 3, \\
{}_9C_2B_1 - {}_9C_4B_2 + {}_9C_6B_3 - {}_9C_8B_4 = \tfrac{7}{2}, & {}_{10}C_2B_1 - {}_{10}C_4B_2 + {}_{10}C_6B_3 - {}_{10}C_8B_4 = 4; \\
\cdots\cdots & \cdots\cdots
\end{array}$$

Since either set will determine B_1, B_2, B_3, $\cdots$ it is obvious that the first n equations in one set and the nth equation of the other set will form a system of $n + 1$ consistent equations in the n quantities B_1, B_2, $\cdots$, B_n; thus the augmented determinant will be zero. For instance, if $n = 4$, we have

[1] Chrystal, *Algebra*, Part II, p. 207 (first edition), or p. 231 (second edition, 1900).

$$\begin{vmatrix} {}_3C_2 & 0 & 0 & 0 & 1 \\ {}_5C_2 & {}_5C_4 & 0 & 0 & 3 \\ {}_7C_2 & {}_7C_4 & {}_7C_6 & 0 & 5 \\ {}_9C_2 & {}_9C_4 & {}_9C_6 & {}_9C_8 & 7 \\ {}_{10}C_2 & {}_{10}C_4 & {}_{10}C_6 & {}_{10}C_8 & 8 \end{vmatrix} = 0, \quad \text{and} \quad \begin{vmatrix} {}_4C_2 & 0 & 0 & 0 & 2 \\ {}_6C_2 & {}_6C_4 & 0 & 0 & 4 \\ {}_8C_2 & {}_8C_4 & {}_8C_6 & 0 & 6 \\ {}_{10}C_2 & {}_{10}C_4 & {}_{10}C_6 & {}_{10}C_8 & 8 \\ {}_9C_2 & {}_9C_4 & {}_9C_6 & {}_9C_8 & 7 \end{vmatrix} = 0,$$

which show at once the scheme for writing out the corresponding determinants of order $n + 1$.

Let us consider the determinant on the left. Subtract three times the last column from the first. The first element in the first column becomes zero, and the other elements take the form

$${}_rC_2 - 3(r - 2) = \frac{r^2 - 7r + 12}{1 \cdot 2} = \frac{(r - 3)(r - 4)}{1 \cdot 2} = {}_{r-3}C_2.$$

When we expand with reference to the first row we obtain

$$\begin{vmatrix} {}_2C_2 & {}_5C_4 & 0 & 0 \\ {}_4C_2 & {}_7C_4 & {}_7C_6 & 0 \\ {}_6C_2 & {}_9C_4 & {}_9C_6 & {}_9C_8 \\ {}_7C_2 & {}_{10}C_4 & {}_{10}C_6 & {}_{10}C_8 \end{vmatrix} = 0.$$

A corresponding reduction of the other determinant can be made.

In the first determinant multiply the first four columns by 2, 4, 6, 8, respectively, and divide the rows by 3, 5, 7, 9, 10, respectively, and we obtain

$$\begin{vmatrix} {}_2C_1 & 0 & 0 & 0 & \frac{1}{3} \\ {}_4C_1 & {}_4C_3 & 0 & 0 & \frac{3}{5} \\ {}_6C_1 & {}_6C_3 & {}_6C_5 & 0 & \frac{5}{7} \\ {}_8C_1 & {}_8C_3 & {}_8C_5 & {}_8C_7 & \frac{7}{9} \\ {}_9C_1 & {}_9C_3 & {}_9C_5 & {}_9C_7 & \frac{8}{10} \end{vmatrix} = 0,$$

a similar transformation being possible for the second determinant.

Various other transformations of the original determinant are of course possible. A direct proof of the vanishing of some of these determinants might be of interest.

II. Relating to the Teaching of Logarithms.

By T. M. Simpson, University of Wisconsin.

The present note is the result of observations made in teaching logarithms to a large number of students. In departing somewhat from current textbook